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A Construction of a Curl-Free but Non-Conservative Vector Field on a Non-Simply Connected Subdomain of $\mathbb{R}^n$

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A Construction of a Curl-Free but Non-Conservative Vector Field on a Non-Simply Connected Subdomain of $\mathbb{R}^n$

Abstract. We explicitly construct a curl-free but non-conservative vector field on the domain $\mathbb{R}^n \setminus (\mathbb{R}^{n-2} \times \{0\} \times [-2,2])$ which is a non-simply connected subspace of $\mathbb{R}^n$. We first establish that $\pi_1(\Omega) \cong \mathbb{Z}$ and identify a generator loop, linking the removed region. We derive a representation for all smooth curl-free vector fields in $\mathbb{R}^n$. We then evaluate the circulation of this field along the linking loop. We explicitly construct functions H and K by integrating a family of fields distributed across the removed region. The resulting vector field is smooth on Ω, and has a non-zero circulation, proving it is non-conservative. The construction provides a concrete realisation of a non-trivial element of $H^1_{dR}(\Omega)$ through *elementary multivariable calculus*.

1 Introduction

One of the most notable results in multivariable calculus, often attributed to Poincaré, states that on a simply connected domain U, every curl-free vector field is conservative [Poincaré 1887]. The reader may therefore infer that a curl-free field and a conservative field are equivalent. However, it turns out this is not the case.

The equivalence depends critically on the topology of the domain. On non-simply connected domains, the topological obstructions may prevent the existence of a global potential despite vanishing curl. A vector field can be curl-free everywhere and yet fail to have a global potential.

Foote [Foote 2018] precisely characterises curl-free but non-conservative vector fields on $\mathbb{R}^2 \setminus \{0\}$, showing that any field differs from a conservative field by a constant multiple of the vector field

$$\mathbf{F} = \frac{-y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}.$$

This field is curl-free away from the origin, but has a non-zero circulation along any circle C centered at the origin.

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 2\pi,$$

and is not exact.

While one may expect to find analogous examples in higher dimensions, this does not extend directly. For instance, no construction would arise from removing a single point in a higher dimension, since that domain would then be simply connected. A key geometric insight is that to obtain a non-trivial fundamental group in $\mathbb{R}^n$, one must remove a set with codimension two. This is the minimum codimension at which loops can wind around the removed set without being contractible. Motivated by this, we consider the domain

$$\mathbb{R}^n \setminus \left(\mathbb{R}^{n-2} \times \{0\} \times [-2,2]\right).$$

This is a higher-dimensional analogue of the punctured plane.

For example, consider the vector field $F = (2x, 2y, 2z)$ in $\mathbb{R}^3$. This field is curl-free and possesses a global potential $\Phi = x^2 + y^2 + z^2$.

There is extensive literature on this phenomenon at the abstract level. The existence of a curl-free but non-conservative vector field is quantified by the first de Rham [de Rham 1984] cohomology group $H^1_{dR}(U)$, which detects differential forms which are closed but not exact. Poincaré's lemma [Poincaré 1887] guarantees its absence on contractible domains. These approaches typically rely on tools such as differential forms, cohomology, and other tools from algebraic topology. In contrast, the objective of the present paper is not to develop the abstract theory, but to make the underlying obstruction visible through elementary methods. Using only tools from multivariable calculus and basic topology, we construct and analyse an explicit example that reveals how the topology of a domain gives rise to curl-free vector fields that fail to be conservative.

Theorem (Main Result)**.** There exists a smooth, curl-free vector field $\mathbf{F}$ on $\Omega = \mathbb{R}^n \setminus (\mathbb{R}^{n-2} \times \{0\} \times [-2,2])$ whose circulation around the generator loop $\gamma(t) = (a_1, \ldots, a_{n-2}, R\sin t, R\cos t)$ equals $-8\pi C$ for any $C \neq 0$. In particular, $\mathbf{F}$ is not conservative.

The argument is structured as follows. We first study the topology of the domain, determine its fundamental group, and identify a generator linking loop that links the removed region. We then derive a general representation of all curl-free fields on Ω, which allows us to reduce the circulation integral to a much simpler expression. Finally, we construct explicit functions H and K by integrating a family of planar singular fields along the removed segment, and show directly that the resulting circulation is $-8\pi C$.

2 Topology of the Domain

In order to determine whether a curl-free field on Ω can be non-conservative, one must understand the topology of Ω. The key question is whether Ω has non-contractible loops. These are loops that cannot be continuously shrunk to a point without leaving the domain. If such a loop does exist, the domain is said to be non-simply connected and this loop becomes our test for non-conservativeness. We show that Ω has fundamental group $\pi_1(\Omega) \cong \mathbb{Z}$, identify an explicit generator loop, and verify that it links the removed set.

We work with a family of domains. For $n \geq 3$ and $0 \leq m \leq n-1$, define

$$\Omega_m := \mathbb{R}^n \setminus (\mathbb{R}^m \times \{0\}^{n-m-1} \times [-2,2]).$$

The parameter m controls how much of $\mathbb{R}^n$ is removed. The interesting case, as we shall see, is $m = n-2$, where the removed set has codimension exactly two. It is the precise condition needed for loops to wind around it non-trivially.

Definition 2.1 (Non-Simply Connected Domains)**.** A domain $U \subset \mathbb{R}^n$ is said to be *non-simply connected* if it is not simply connected; that is, if there exists a continuous loop $\gamma : [0,1] \to U$ with $\gamma(0) = \gamma(1)$ such that there is no continuous homotopy $H : [0,1] \times [0,1] \to U$ satisfying

$$H(s,0) = \gamma(s), \qquad H(s,1) = p \quad \text{for some } p \in U,$$

while keeping the endpoints fixed, i.e. $H(0,t) = H(1,t) = p$.

Lemma 2.2 (Computing $\pi_1(\Omega_m)$)**.** *Let $n \geq 3$ and let $\mathbb{R}^m \subset \mathbb{R}^n$ be embedded as a linear subspace. Define*

$$\Omega_m := \mathbb{R}^n \setminus (\mathbb{R}^m \times \{0\}^{n-m-1} \times [-2,2]).$$

Then:

1. *If $m \leq n-3$, the domain Ω_m is simply connected.*
2. *If $m = n-2$, then $\pi_1(\Omega_m) \cong \mathbb{Z}$.*
3. *If $m = n-1$, the domain Ω_m is disconnected.*

The three cases have a clean geometric interpretation. When too little is removed (Case 1), loops have enough room to slip past the removed set and contract to a point. When too much is removed (Case 3), the domain falls apart into two pieces, and loops cannot even reach across. Only at the second case does the interesting part happen. Loops wind around the removed set in a way that cannot be undone.

Proof. Since all m-dimensional linear subspaces of $\mathbb{R}^n$ are related by linear isomorphisms, and linear isomorphisms of $\mathbb{R}^n$ are homeomorphisms, we may assume without

loss of generality that

$$\mathbb{R}^m = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_{m+1} = x_{m+2} = \cdots = x_n = 0\}.$$

Thus,

$$\Omega_m = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid (x_{m+1}, \ldots, x_{n-1}) \neq 0 \text{ or } x_n \notin [-2, 2]\}.$$

The first step is to separate the coordinates that interact with the removed set from those that do not. Define the map

$$\Phi : \Omega_m \longrightarrow \Big((\mathbb{R}^{n-m} \setminus \{0\}) \cup (\mathbb{R}^{n-m-1} \times (\mathbb{R} \setminus [-2, 2]))\Big) \times \mathbb{R}^m$$

by

$$\Phi(x_1, \ldots, x_n) = \Big((x_{m+1}, \ldots, x_n), (x_1, \ldots, x_m)\Big).$$

This map is bijective, continuous, and has a continuous inverse given by concatenation of coordinates. Therefore, Φ is a homeomorphism, so

$$\Omega_m \cong \Big((\mathbb{R}^{n-m} \setminus \{0\}) \cup (\mathbb{R}^{n-m-1} \times (\mathbb{R} \setminus [-2, 2]))\Big) \times \mathbb{R}^m.$$

The first factor captures all the topological complexity of Ω_m. The second factor $\mathbb{R}^m$ is contractible and will contribute nothing to the fundamental group.

$$(\mathbb{R}^{n-m} \setminus \{0\}) \cup (\mathbb{R}^{n-m-1} \times (\mathbb{R} \setminus [-2, 2])) = \mathbb{R}^k \setminus (\{0\}^{k-1} \times [-2, 2]), \quad \text{where } k = n - m.$$

This space deformation retracts onto $\mathbb{R}^k \setminus \{0\}$. As a result,

$$\Omega_m \simeq (\mathbb{R}^k \setminus \{0\}) \times \mathbb{R}^m.$$

Applying the product formula for fundamental groups, and using the fact that $\mathbb{R}^m$ is contractible

$$\pi_1(\Omega_m) \cong \pi_1(\mathbb{R}^k \setminus \{0\}), \quad k = n - m.$$

It remains to compute $\pi_1(\mathbb{R}^k \setminus \{0\})$. The space $\mathbb{R}^k \setminus \{0\}$ deformation retracts onto the unit sphere

$$\mathrm{S}^{k-1} = \{x \in \mathbb{R}^k : \|x\| = 1\}$$

via the radial homotopy

$$\mathrm{H}(x, t) = (1 - t)x + t\frac{x}{\|x\|}.$$

Thus, $\mathbb{R}^k \setminus \{0\}$ and S^{k-1} are homotopy equivalent.

The three cases now follow from standard facts about spheres and $\pi_1(\mathrm{S}^{k-1})$.

Case 1: $k \geq 3$**.** If $k \geq 3$, then $k - 1 \geq 2$, and the sphere S^{k-1} is simply connected. This implies that

$$\pi_1(\mathbb{R}^k \setminus \{0\}) = 0,$$

so Ω_m is simply connected whenever $m \leq n - 3$. Loops in this regime have enough room

to slip past the removed set entirely.

Case 2: $k = 2$**.** This is the case that matters for our construction. Here

$$\mathbb{R}^2 \setminus \{0\} \simeq S^1,$$

whose fundamental group is well known [Hatcher 2002]:

$$\pi_1(S^1) \cong \mathbb{Z}.$$

Consequently,

$$\pi_1(\Omega_{n-2}) \cong \mathbb{Z}.$$

The domain is path-connected but not simply connected, and its loops are classified by how many times they wind around the removed set.

Case 3: $k = 1$ **(i.e.** $m = n-1$**).** Here

$$\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty),$$

which is disconnected. Using the homeomorphism above,

$$\Omega_{n-1} \cong (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^{n-1}.$$

Since $\mathbb{R} \setminus \{0\}$ is disconnected, it follows that Ω_{n-1} is disconnected too. Consequently, any loop in Ω_{n-1} must lie entirely within a single connected component. Therefore, every loop in Ω_{n-1} is null-homotopic within its component. □

Thus, we restrict our attention to the case $k = 2$, in which the complement $\mathbb{R}^n \setminus (\mathbb{R}^{n-2} \times \{0\} \times [-2, 2])$ is path-connected but not simply connected. When $n = 3$, this geometrically corresponds to removing a thick line from the domain in the xz-plane. The Figure 1 illustrates the situation.

Since $\pi_1(\Omega) \cong \mathbb{Z}$, a curl-free field $\mathbf{F}$ on Ω is conservative if and only if its circulation vanishes around a single generator of this group. Everything reduces to finding such a generator explicitly and evaluating the circulation around it.

We must choose a linking loop that lies entirely in $\Omega = \mathbb{R}^n \setminus (\mathbb{R}^{n-2} \times \{0\} \times [-2, 2])$ and links the removed set $\mathbb{R}^{n-2} \times \{0\} \times [-2, 2]$. We choose the loop $\gamma(t) : [0, 2\pi] \to \mathbb{R}^n$ defined by

$$\gamma(t) = (a_1, a_2, a_3, \ldots, a_{n-2}, R\sin t, R\cos t),$$

for some constants $a_1, a_2, a_3, \ldots, a_{n-2} \in \mathbb{R}$ and a radius $R > 2$.

Geometrically, this is a circle of radius R in the (x_{n-1}, x_n)-plane, with the first $n-2$ coordinates held fixed. We verify two things: that γ lies in Ω, and that it represents the generator of $\pi_1(\Omega)$.

First, we need to verify whether the loop

$$\gamma(t) = (a_1, a_2, a_3, \ldots, a_{n-2}, R\sin t, R\cos t)$$

is a valid linking loop for the domain, that is, it lies entirely in Ω and it links around $\mathbb{R}^{n-2} \times \{0\} \times [-2,2]$.

We first show that $\gamma(t) \in \Omega$. For each $t \in [0, 2\pi]$, the last two coordinates of $\gamma(t)$ are

$$(x_{n-1}(t), x_n(t)) = (\mathrm{R}\sin t, \mathrm{R}\cos t).$$

For a point to lie in the removed set,

$$x_{n-1} = 0 \quad \text{and } x_n \in [-2,2].$$

Thus, if $\gamma(t)$ were in the removed set, then we would have

$$\mathrm{R}\sin t = 0 \quad \text{and } \mathrm{R}\cos t \in [-2,2].$$

This implies that

$$\mathrm{R}\cos t = \pm \mathrm{R} \notin [-2,2]$$

as $\mathrm{R} > 2$. Thus, $\gamma(t) \in \Omega$.

Secondly, we now prove that $\gamma(t)$ links the removed set by constructing an explicit deformation retraction.

Write a point of $\mathbb{R}^n$ as (x', y, z), where $x' = (x_1, \ldots, x_{n-2}) \in \mathbb{R}^{n-2}$ and $(y, z) \in \mathbb{R}^2$. Then

$$\Omega = \{(x', y, z) \in \mathbb{R}^n : y \neq 0 \text{ or } z \notin [-2,2]\}.$$

Define

$$\mathrm{H} : \Omega \times [0,1] \to \Omega$$

by

$$\mathrm{H}((x', y, z), s) = ((1-s)x', y, z).$$

This homotopy continuously collapses the first $n-2$ coordinates to zero while leaving y and z unchanged. Since a point exits Ω only when $y = 0$ and $z \in [-2,2]$, and H never alters y or z, the image stays in Ω throughout.

We verify the three conditions for a deformation retraction onto

$$\Omega' = \{(0, \ldots, 0, y, z) \in \mathbb{R}^n : y \neq 0 \text{ or } z \notin [-2,2]\}.$$

Note that H is continuous. Moreover,

$$\mathrm{H}((x', y, z), 0) = (x', y, z),$$

so $\mathrm{H}(\cdot, 0)$ is the identity on Ω. Furthermore,

$$\mathrm{H}((x', y, z), 1) = (0, y, z) \in \Omega'.$$

If $(x', y, z) \in \Omega'$, then $x' = 0$, which implies that

$$\mathrm{H}((0, y, z), s) = (0, y, z)$$

for all s, so the points of Ω' remain fixed. Finally, since $y \neq 0$ or $z \notin [-2,2]$ for all points of Ω, the homotopy never enters the removed set. Thus, H is a deformation retraction of Ω onto Ω'.

The space Ω' is homeomorphic to $\mathbb{R}^2 \setminus \{0\}$, and therefore

$$\pi_1(\Omega) \cong \pi_1(\mathbb{R}^2 \setminus \{0\}) \cong \mathbb{Z}.$$

Applying homotopy to $\gamma(t)$, we obtain

$$\mathrm{H}(\gamma(t), s) = ((1-s)a_1, \ldots, (1-s)a_{n-2}, \mathrm{R}\sin t, \mathrm{R}\cos t).$$

At $s = 1$,

$$\mathrm{H}(\gamma(t), 1) = (0, \ldots, 0, \mathrm{R}\sin t, \mathrm{R}\cos t),$$

which is the standard circle in $\mathbb{R}^2 \setminus \{0\}$ and represents a generator of its fundamental group.

Since deformation retractions induce isomorphisms on fundamental groups, it follows that $\gamma(t)$ is not null-homotopic in Ω. Therefore, $\gamma(t)$ links the removed set $\mathbb{R}^{n-2} \times \{0\} \times [-2,2]$.

3 Structure of Curl-Free Vector Fields

Having established that $\pi_1(\Omega) \cong \mathbb{Z}$ and identified the generator loop γ, we know that non-conservativeness of a curl-free field $\mathbf{F}$ reduces to showing that its circulation around γ is nonzero. To compute this circulation, we first need to understand what a curl-free field looks like structurally. The lemma in this section shows that every component F_i of a curl-free field is determined, up to a function of the remaining variables, by integrating the first component F_1. These residual functions are precisely what the circulation will depend on, as we show in Section 4.

We now derive a structural representation of curl-free vector fields in $\mathbb{R}^n$.

We work with the differential 1-form associated to $\mathbf{F}$, defined by $\omega = \sum_{i=1}^{n} \mathrm{F}_i \, dx_i$. A vector field $\mathbf{F}$ is *curl-free* if the exterior derivative of ω vanishes, which in coordinates means

$$\frac{\partial \mathrm{F}_i}{\partial x_j} = \frac{\partial \mathrm{F}_j}{\partial x_i} \quad \text{for all } 1 \leq i, j \leq n.$$

This is the condition we work with throughout. We say $\mathrm{U} \subset \mathbb{R}^n$ is x_1-convex with respect to $x_1 = s_0$ if for every point $(x_1, \ldots, x_n) \in \mathrm{U}$, the entire segment $\{(t, x_2, \ldots, x_n) : t \text{ between } s_0 \text{ and } x_1\}$ lies in U. This is the condition that allows us to integrate in the x_1-direction without leaving the domain.

Lemma 3.1 (Characterization of curl-free vector fields in $\mathbb{R}^n$)**.** *Let* $\mathrm{U} \subset \mathbb{R}^n$ *be open and*

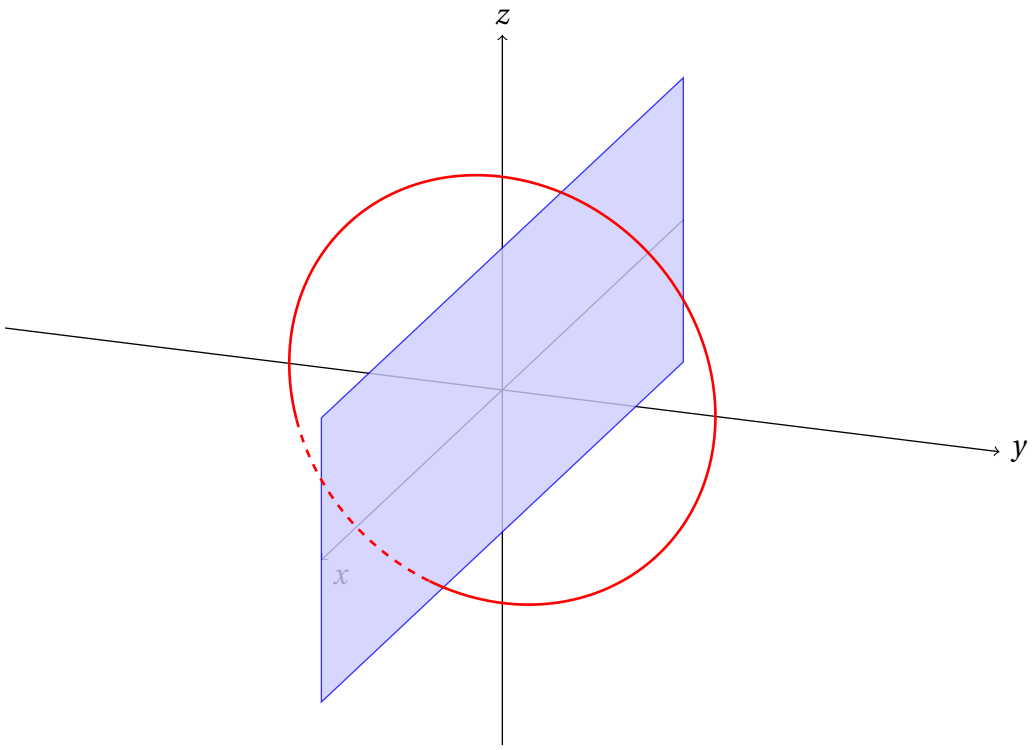


Figure 1: The case $n = 3$: the blue strip is the removed set $\mathbb{R}^{n-2} \times \{0\} \times [-2,2]$ in the xz-plane, and the red circle is the generator loop γ, which winds once around the strip and cannot be contracted to a point within Ω.

x_1-convex with respect to the hyperplane $x_1 = s_0$. Let

$$\mathbf{F} = (\mathrm{F}_1, \mathrm{F}_2, \ldots, \mathrm{F}_n)$$

be a C^1 vector field on U, and assume $\mathrm{F}_1 \in \mathrm{C}^2$.

Then the associated 1-form

$$\omega = \sum_{i=1}^{n} \mathrm{F}_i \, dx_i$$

is closed (i.e. $d\omega = 0$) if and only if for each $i = 2, \ldots, n$ there exists a function

$$\mathrm{H}_i \in \mathrm{C}^1(\mathbb{R}^{n-1})$$

such that

$$\mathrm{F}_i(x_1, \ldots, x_n) = \int_{s_0}^{x_1} \frac{\partial \mathrm{F}_1}{\partial x_i}(u, x_2, \ldots, x_n) \, du + \mathrm{H}_i(x_2, \ldots, x_n),$$

with the compatibility conditions

$$\frac{\partial H_i}{\partial x_j} = \frac{\partial H_j}{\partial x_i} \quad \textit{for all } 2 \le i, j \le n.$$

Proof. ($\Rightarrow$) Assume **F** is closed. In coordinates, this means

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i} \quad \text{for all } 1 \le i, j \le n.$$

The strategy is to integrate this condition in the x_1 direction, recovering each F_i from F_1 up to a function of the remaining variables. Fix an index $i \in \{2, \dots, n\}$. From the closedness condition with $j = 1$, we obtain

$$\frac{\partial F_i}{\partial x_1} = \frac{\partial F_1}{\partial x_i}.$$

Since U is x_1-convex with respect to $x_1 = s_0$, for each fixed $(x_2, \dots, x_n)$ the segment

$$[s_0, x_1] \times \{x_2, x_3, \dots, x_{n-2}, x_{n-1}, x_n\} \subset U$$

lies entirely in the domain. We may therefore integrate the above identity with respect to x_1 along this segment, obtaining

$$F_i(x_1, \dots, x_n) = \int_{s_0}^{x_1} \frac{\partial F_1}{\partial x_i}(u, x_2, \dots, x_n)\, du + H_i(x_2, \dots, x_n),$$

where $H_i(x_2, \dots, x_n)$ is the constant of integration, depending only on the variables held fixed.

It remains to determine the relations among the functions H_i. Fix indices $i, j \in \{2, \dots, n\}$. Differentiating the above representation with respect to x_j gives

$$\frac{\partial F_i}{\partial x_j} = \int_{s_0}^{x_1} \frac{\partial^2 F_1}{\partial x_j \partial x_i}(u, x_2, \dots, x_n)\, du + \frac{\partial H_i}{\partial x_j}.$$

Similarly, interchanging i and j,

$$\frac{\partial F_j}{\partial x_i} = \int_{s_0}^{x_1} \frac{\partial^2 F_1}{\partial x_i \partial x_j}(u, x_2, \dots, x_n)\, du + \frac{\partial H_j}{\partial x_i}.$$

Since $F_1 \in C^2$, Clairaut's theorem [Apostol 1969] implies

$$\frac{\partial^2 F_1}{\partial x_j \partial x_i} = \frac{\partial^2 F_1}{\partial x_i \partial x_j},$$

and hence the integral terms cancel. The curl-free condition then concludes

$$\frac{\partial H_i}{\partial x_j} = \frac{\partial H_j}{\partial x_i} \quad \text{for all } 2 \le i, j \le n.$$

The curl-free condition among the transverse components reduces entirely to the H_i.

($\Leftarrow$) Conversely, suppose that for each $i = 2, \dots, n$ the functions F_i admit representations

$$F_i(x_1, \dots, x_n) = \int_{s_0}^{x_1} \frac{\partial F_1}{\partial x_i}(u, x_2, \dots, x_n)\, du + H_i(x_2, \dots, x_n),$$

with $H_i \in C^1$ satisfying

$$\frac{\partial H_i}{\partial x_j} = \frac{\partial H_j}{\partial x_i} \quad \text{for all } 2 \le i, j \le n.$$

We verify that $\mathbf{F}$ is curl-free. Differentiating with respect to x_1 yields

$$\frac{\partial F_i}{\partial x_1} = \frac{\partial F_1}{\partial x_i} \quad \text{for } i = 2, \dots, n.$$

Next, for $i, j \ge 2$, differentiating with respect to x_j and using $F_1 \in C^2$ gives

$$\frac{\partial F_i}{\partial x_j} = \int_{s_0}^{x_1} \frac{\partial^2 F_1}{\partial x_j \partial x_i}(u, x_2, \dots, x_n)\, du + \frac{\partial H_i}{\partial x_j},$$

and similarly for $\frac{\partial F_j}{\partial x_i}$. By Clairaut's theorem [Apostol 1969] of mixed partial derivatives and the compatibility conditions on the H_i, we obtain

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}.$$

Thus all mixed partial derivatives satisfy

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i} \quad \text{for all } 1 \le i, j \le n,$$

and hence $d\omega = 0$. This completes the proof. □

In other words, knowing F_1 and the integration functions $H_2, \dots, H_n$ completely determines the curl-free field $\mathbf{F}$. The compatibility conditions on the H_i are exactly what remains of the curl-free condition once F_1 has been fixed. As we show in Section 4, the circulation of $\mathbf{F}$ around γ turns out to depend only on H_{n-1} and H_n — the two integration functions corresponding to the last two coordinates, which are the ones that vary along γ.

4 Reduction of the Circulation Integral

The previous section showed that every curl-free field $\mathbf{F}$ on Ω is determined by a base component F_1 and integration functions H_i. This section uses that structure to simplify the conservativeness test. Recall from Section 2 that $\pi_1(\Omega) \cong \mathbb{Z}$, so a curl-free field is conservative if and only if its circulation around the generator γ vanishes. The key result of this section is that this circulation depends only on the integration functions H_{n-1}

and H_n. This means we can choose F_1 freely and focus the entire construction on H_{n-1} and H_n.

A generator of $\pi_1(\Omega)$ is represented by the loop $\gamma(t) = (a_1, a_2, a_3, \dots, a_{n-2}, R\sin t, R\cos t)$. This loop winds once around the origin and therefore represents the generator of $\pi_1(\mathbb{R}^2 \setminus \{0\}) \cong \mathbb{Z}$. Therefore, to check whether a curl-free field $\mathbf{F}$ is conservative, it suffices to compute $\oint_\gamma \mathbf{F} \cdot d\mathbf{r}$. If this integral vanishes, then it vanishes for all closed loops in Ω.

Lemma 4.1. *The circulation of a curl-free field* $\mathbf{F}$ *around* γ *depends only on the integration functions* H_{n-1} *and* H_n*, and not on the base component* F_1*. In particular,* $\mathbf{F}$ *is non-conservative if and only if the two-dimensional circulation integral*

$$I_{HK} = \oint_\gamma \big(H(y,z)\,dy + K(y,z)\,dz\big)$$

is nonzero, where $H = H_{n-1}$ *and* $K = H_n$ *restricted to the last two coordinates.*

Proof. For concreteness we take $a_i = 2$ for $i = 1, \dots, n-2$ and $R = 3$, so that

$$\gamma(t) = (2, 2, \dots, 2, 3\sin t, 3\cos t),$$

with $\gamma'(t) = (0, 0, \dots, 0, 3\cos t, -3\sin t)$. Thus $\gamma'(t)$ has non-zero components only in the last two coordinates. The argument works for any valid a_i and $R > 2$ by the same calculation.

Evaluating the vector field along the loop gives

$$\mathbf{F}(\gamma(t)) = \big(F_1(\gamma(t)), \dots, F_{n-2}(\gamma(t)), F_{n-1}(\gamma(t)), F_n(\gamma(t))\big).$$

Taking the dot product, all contributions from the first $n-2$ components vanish, and we obtain

$$\mathbf{F}(\gamma(t)) \cdot \gamma'(t) = (3\cos t)\,F_{n-1}(\gamma(t)) - (3\sin t)\,F_n(\gamma(t)).$$

We denote the last two coordinates by

$$y := x_{n-1}, \qquad z := x_n.$$

Fix any $s_0 \in \mathbb{R}$ such that the segment $[s_0, 2] \times \{2, \dots, 2, 3\sin t, 3\cos t\}$ lies in Ω. Since the domain is x_1-convex, there must exist such s_0. Applying the representation theorem, the line integral becomes

$$\begin{aligned} I = \int_0^{2\pi} \Bigg[& 3\cos t \left(\int_{s_0}^{2} f_y(u, 2, \dots, 2, 3\sin t, 3\cos t)\,du + H(2, \dots, 2, 3\sin t, 3\cos t) \right) \\ & - 3\sin t \left(\int_{s_0}^{2} f_z(u, 2, \dots, 2, 3\sin t, 3\cos t)\,du + K(2, \dots, 2, 3\sin t, 3\cos t) \right) \Bigg] dt, \end{aligned}$$

where $f_y = \frac{\partial f}{\partial y}$ and $f_z = \frac{\partial f}{\partial z}$.

Let $f = \mathrm{F}_1$. Now consider the function

$$t \longmapsto f(u, 2, \ldots, 2, 3\sin t, 3\cos t),$$

with u treated as a parameter. By the chain rule,

$$\frac{d}{dt} f(u, 2, \ldots, 2, 3\sin t, 3\cos t) = \frac{\partial f}{\partial y}(u, 2, \ldots, 2, 3\sin t, 3\cos t) \cdot 3\cos t$$

$$- \frac{\partial f}{\partial z}(u, 2, \ldots, 2, 3\sin t, 3\cos t) \cdot 3\sin t.$$

Integrating both sides with respect to u from s_0 to x, with t fixed, yields

$$\int_{s_0}^{x} \frac{d}{dt} f(u, 2, \ldots, 2, 3\sin t, 3\cos t)\, du = 3\cos t \int_{s_0}^{x} f_y(u, 2, \ldots, 2, 3\sin t, 3\cos t)\, du$$
$$- 3\sin t \int_{s_0}^{x} f_z(u, 2, \ldots, 2, 3\sin t, 3\cos t)\, du.$$

Since differentiation with respect to t and integration with respect to u commute under the given smoothness assumptions, we obtain

$$\int_{s_0}^{x} \frac{d}{dt} f(u, 2, \ldots, 2, 3\sin t, 3\cos t)\, du = \frac{d}{dt} \int_{s_0}^{x} f(u, 2, \ldots, 2, 3\sin t, 3\cos t)\, du. \tag{1}$$

Rearranging our original integral, we get

$$\begin{aligned} \mathrm{I} = \int_0^{2\pi} \Bigg(& 3\cos t \left(\int_{s_0}^{x} f_y(u, 2, \ldots, 2, 3\sin t, 3\cos t)\, du \Big|_{x=2} \right) \\ & - 3\sin t \left(\int_{s_0}^{x} f_z(u, 2, \ldots, 2, 3\sin t, 3\cos t)\, du \Big|_{x=2} \right) \Bigg) dt \\ & + \int_0^{2\pi} \Big(3\cos t\, \mathrm{H}(2, \ldots, 2, 3\sin t, 3\cos t) \; - \; 3\sin t\, \mathrm{K}(2, \ldots, 2, 3\sin t, 3\cos t) \Big)\, dt. \end{aligned}$$

Using the result in (1), we get,

$$\begin{aligned} \mathrm{I} = & \int_0^{2\pi} \frac{d}{dt} \left(\int_{s_0}^{x} f(u, 2, \ldots, 2, 3\sin t, 3\cos t)\, du \right) dt \\ & + \int_0^{2\pi} \left(3\cos t \cdot \mathrm{H}(2, \ldots, 2, 3\sin t, 3\cos t) \; - \; 3\sin t \cdot \mathrm{K}(2, \ldots, 2, 3\sin t, 3\cos t) \right) dt. \end{aligned}$$

The first part of this integral is

$$\int_0^{2\pi} \frac{d}{dt} \left(\int_{s_0}^{x} f(u, 2, \ldots, 2, 3\sin t, 3\cos t)\, du \right) dt.$$

The integrand is a total derivative with respect to t, so by the Fundamental Theorem of

Calculus, it evaluates to

$$\int_{s_0}^{x} f(u,2,\ldots,2,3\sin 2\pi,3\cos 2\pi)\,du \;-\; \int_{s_0}^{x} f(u,2,\ldots,2,3\sin 0,3\cos 0)\,du = 0.$$

Since the F_1 dependent terms have vanished, what remains is only dependent on H and K. Therefore, the integral reduces to

$$I_{HK} = \int_0^{2\pi} (\,3\cos t\cdot H(2,\ldots,2,3\sin t,3\cos t) \;-\; 3\sin t\cdot K(2,\ldots,2,3\sin t,3\cos t))\;dt.$$

Along the loop γ, the last two coordinates are by definition

$$y(t) = 3\sin t, \quad z(t) = 3\cos t, \quad 0 \le t \le 2\pi.$$

Then,

$$y'(t) = 3\cos t \qquad z'(t) = -3\sin t.$$

Writing $H(y,z)$ and $K(y,z)$ for the restrictions of H and K to the slice where the first $n-2$ coordinates equal 2, that is, $H(y,z) := H(2,\ldots,2,y,z)$ and $K(y,z) := K(2,\ldots,2,y,z)$. The integral becomes

$$\int_0^{2\pi} \big(H(y,z)\,y'(t) + K(y,z)\,z'(t)\big)\,dt.$$

This is precisely the line integral of the two-dimensional field (H,K) around the projection of γ onto the (y,z)-plane

$$I_{HK} = \oint_{\Upsilon} \big(H(y,z)\,dy + K(y,z)\,dz\big).$$

□

The reduction is complete. To show that **F** is non-conservative, it now suffices to construct functions H and K satisfying the curl-free compatibility condition $K_y = H_z$ on Ω, for which the two-dimensional circulation integral I_{HK} is nonzero. The analysis of Sections 5 and 6 will confirm that such functions exist and produce a nonzero circulation.

5 Constructing the Integration Functions

We now turn to the heart of the construction. Recall from the previous section that the conservativeness now depends entirely on whether the circulation integral

$$I_{HK} = \oint_{\Upsilon} \big(H(y,z)\,dy + K(y,z)\,dz\big)$$

vanishes. We need to construct functions H and K that satisfy the curl-free compatibility condition $K_y = H_z$ on Ω, and for which the circulation is non-zero.

The construction represents the vector field as arising from a continuous distribution of infinitesimal singularities along the line segment $y_0 \in [-2,2]$. The intent is to produce a field that is curl-free on Ω but singular along the removed strip, so that the circulation around a loop γ linking the strip once is obtained by summing the contributions from all such singularities. This is what ultimately results in a non-zero circulation.

To allow sufficient generality, we begin with the following ansatz. The variable y_0 represents the coordinate along the removed strip, while (y,z) denotes the point at which the field is evaluated.

$$h(y,z;y_0) = \frac{\mathrm{A}(y-y_0)+\mathrm{B}z}{(y-y_0)^2+z^2} \quad \text{and} \quad k(y,z;y_0) = \frac{\mathrm{C}(y-y_0)+\mathrm{D}z}{(y-y_0)^2+z^2}$$

The constants A, B, C, and D are to be determined. The denominator $(y-y_0)^2+z^2$ is the squared distance from the point (y,z) to the source $(y_0,0)$. This vanishes precisely when $(y,z)=(y_0,0)$, which is essentially along the removed strip. However, as we will show later, for (y,z) in our domain Ω, the integrand is a continuous function of y_0 on the interval $[-2,2]$. Hence, the integrals are well-defined as ordinary Riemann integrals.

We now impose the curl-free condition.

$$\frac{\partial \mathrm{K}}{\partial y} = \frac{\partial \mathrm{H}}{\partial z},$$

The integrands h and k must satisfy the same condition pointwise. Computing the partial derivatives gives

$$\frac{((y-y_0)^2+z^2)\,\mathrm{C} - (\mathrm{C}(y-y_0)+\mathrm{D}z)\,2(y-y_0)}{((y-y_0)^2+z^2)^2} =$$

$$\frac{((y-y_0)^2+z^2)\,\mathrm{B} - (\mathrm{A}(y-y_0)+\mathrm{B}z)\,2z}{((y-y_0)^2+z^2)^2}.$$

Expanding the numerators yields

$$\frac{\mathrm{C}z^2 - \mathrm{C}(y-y_0)^2 - 2\mathrm{D}(y-y_0)z}{((y-y_0)^2+z^2)^2} = \frac{-\mathrm{B}z^2 + \mathrm{B}(y-y_0)^2 - 2\mathrm{A}(y-y_0)z}{((y-y_0)^2+z^2)^2}.$$

Equating the coefficients of $(y-y_0)^2$, z^2, and $z(y-y_0)$ on both sides gives us that

$$\mathrm{B} = -\mathrm{C} \quad \text{and} \quad \mathrm{A} = \mathrm{D}.$$

With these constraints in place, the two free parameters A and C remain, and the integrand fields take the form

$$h(y,z;y_0) = \frac{\mathrm{A}(y-y_0)-\mathrm{C}z}{(y-y_0)^2+z^2}, \qquad k(y,z;y_0) = \frac{\mathrm{C}(y-y_0)+\mathrm{A}z}{(y-y_0)^2+z^2},$$

for a fixed y_0. Thus, for every y_0,

$$k_y(y,z;y_0) = h_z(y,z;y_0).$$

Integrating both sides over $y_0 \in [-2,2]$, and using the fact that the equality holds pointwise, we obtain

$$\mathrm{K}_y(y,z) = \mathrm{H}_z(y,z).$$

We therefore define the following constructions for H and K:

$$\mathrm{H}(y,z) = \int_{-2}^{2} \frac{\mathrm{A}(y-y_0) - \mathrm{C}z}{(y-y_0)^2 + z^2}\, dy_0,$$

$$\mathrm{K}(y,z) = \int_{-2}^{2} \frac{\mathrm{C}(y-y_0) + \mathrm{A}z}{(y-y_0)^2 + z^2}\, dy_0.$$

Hence, the curl-free condition is satisfied by this construction. With the construction in place, we turn in the next section to the explicit evaluation of I_{HK}, which will confirm that the field is genuinely non-conservative.

6 Evaluation of the Circulation

We now explicitly evaluate the circulation I_{HK}. The strategy has two steps. We first substitute the construction of H and K into the integral and split the result according to the constants A and C. A symmetry argument then shows that the A component vanishes identically, reducing the evaluation to a single integral in C. We compute this explicitly, arriving at $\mathrm{I}_{\mathrm{HK}} = -8\pi\mathrm{C} \neq 0$.

Lemma 6.1. *For the functions* H *and* K *defined in Section 5 with the parameter* $\mathrm{C} \neq 0$*, the circulation integral satisfies* $\mathrm{I}_{\mathrm{HK}} = -8\pi\mathrm{C}$.

Proof. Parameterising γ by $y(t) = 3\sin t$, $z(t) = 3\cos t$ for $t \in [0, 2\pi]$. This leads to

$$\mathrm{I}_{\mathrm{HK}} = \oint_{\gamma} (\mathrm{H}\, dy + \mathrm{K}\, dz) = \int_0^{2\pi} (\mathrm{H}(3\sin t, 3\cos t)\, y'(t) + \mathrm{K}(3\sin t, 3\cos t)\, z'(t))\, dt$$

But since $y'(t) = 3\cos t$ and $z'(t) = -3\sin t$, we obtain

$$\mathrm{I}_{\mathrm{HK}} = 3\int_0^{2\pi} (\mathrm{H}(3\sin t, 3\cos t)\cos t - \mathrm{K}(3\sin t, 3\cos t)\sin t)\, dt$$

Substituting our constructions of H and K,

$$\mathrm{I_{HK}} = 3\int_0^{2\pi}\int_{-2}^{2}\left(\frac{\mathrm{A}(3\sin t - y_0) - \mathrm{C}(3\cos t)}{(3\sin t - y_0)^2 + (3\cos t)^2}\cos t - \frac{\mathrm{C}(3\sin t - y_0) + \mathrm{A}(3\cos t)}{(3\sin t - y_0)^2 + (3\cos t)^2}\sin t\right) dy_0\, dt$$

Grouping terms by the constants A and C, we obtain

$$\mathrm{I_{HK}} = 3\int_0^{2\pi}\int_{-2}^{2}\left(\frac{\mathrm{A}(3\sin t - y_0)(\cos t) - \mathrm{A}(3\cos t)(\sin t)}{(3\sin t - y_0)^2 + (3\cos t)^2} + \frac{-\mathrm{C}(3\sin t - y_0)(\sin t) - \mathrm{C}(3\cos t)(\cos t)}{(3\sin t - y_0)^2 + (3\cos t)^2} dy_0\right) dt$$

which simplifies to

$$\mathrm{I_{HK}} = 3\mathrm{A}\mathrm{I_A} + 3\mathrm{C}\mathrm{I_C}$$

where

$$\begin{aligned}\mathrm{I_A} &= \int_0^{2\pi}\int_{-2}^{2}\frac{(3\sin t - y_0)\cos t - (3\cos t)\sin t}{(3\sin t - y_0)^2 + (3\cos t)^2} dy_0\, dt \\ &= \int_0^{2\pi}\int_{-2}^{2}\frac{-y_0\cos t}{(3\sin t - y_0)^2 + (3\cos t)^2} dy_0\, dt,\end{aligned}$$

$$\begin{aligned}\mathrm{I_C} &= \int_0^{2\pi}\int_{-2}^{2}\frac{-(3\cos t)\cos t - (3\sin t - y_0)\sin t}{(3\sin t - y_0)^2 + (3\cos t)^2} dy_0\, dt \\ &= \int_0^{2\pi}\int_{-2}^{2}\frac{-3 + y_0\sin t}{(3\sin t - y_0)^2 + (3\cos t)^2} dy_0\, dt.\end{aligned}$$

We will be employing a symmetry argument to prove that $\mathrm{I_A} = 0$. We begin by isolating the inner integral over y_0 by defining

$$\mathrm{G}(t) = \int_{-2}^{2}\frac{-y_0\cos t}{(3\sin t - y_0)^2 + (3\cos t)^2} dy_0.$$

The substitution $t \mapsto \pi - t$ is a natural candidate as $\sin(\pi - t) = \sin t$ and $\cos(\pi - t) = -\cos t$. Also the substitution maps $[0, 2\pi]$ to $[-\pi, \pi]$ itself. This means it leaves the denominator of G unchanged while flipping the sign of the numerator. This makes $\mathrm{G}(t)$ antisymmetric.

$$\begin{aligned}\mathrm{G}(\pi - t) &= \int_{-2}^{2}\frac{-y_0\cos(\pi - t)}{(3\sin(\pi - t) - y_0)^2 + (3\cos(\pi - t))^2} dy_0 \\ &= \int_{-2}^{2}\frac{y_0\cos t}{(3\sin t - y_0)^2 + (3\cos t)^2} dy_0 = -\mathrm{G}(t)\end{aligned}$$

Since both $\sin t$ and $\cos t$ are 2π periodic, so is G.

$$I_A = \int_0^{2\pi} G(t)\, dt = \int_0^{2\pi} G(\pi - t)\, dt = \int_0^{2\pi} -G(t)\, dt$$

$$\int_0^{2\pi} G(t)\, dt = -\int_0^{2\pi} G(t)\, dt$$

$$2\int_0^{2\pi} G(t)\, dt = 0$$

Hence, $2I_A = 0$, so $I_A = 0$. This means that

$$I_{HK} = 3AI_A + 3CI_C = 3CI_C.$$

We now compute I_C. We have

$$I_C = \int_0^{2\pi} \left(\int_{-2}^{2} \frac{\sin t\, y_0 - 3}{(3\sin t - y_0)^2 + (3\cos t)^2}\, dy_0 \right) dt.$$

To justify the use of Fubini's theorem, we verify that the integrand belongs to $L^1([-2,2] \times [0,2\pi])$. The denominator

$$(3\sin t - y_0)^2 + (3\cos t)^2$$

is a sum of squares and therefore nonnegative. Equality can occur only if

$$3\cos t = 0 \quad \text{and} \quad 3\sin t - y_0 = 0.$$

The first condition implies that $t = \frac{\pi}{2}, \frac{3\pi}{2}$. At these values, $\sin t = \pm 1$, which would require $y_0 = \pm 3$. Since $y_0 \in [-2,2]$, this cannot occur.

Hence, the denominator is strictly positive on $[-2,2] \times [0,2\pi]$, and the integrand is continuous on this compact set. This confirms that a bound exists, but to apply Fubini's, we must make the L^1 bound explicit, rather than merely asserting it. We establish a quantitative lower bound for the denominator. Expanding the denominator we have,

$$(3\sin t - y_0)^2 + (3\cos t)^2 = 9 - 6y_0 \sin t + y_0^2.$$

For fixed y_0, the only t-dependent term is $-6y_0 \sin t$. Since $\sin t \in [-1,1]$, we have

$$-6y_0 \sin t \geq -6|y_0|.$$

Therefore,

$$9 - 6y_0 \sin t + y_0^2 \geq 9 - 6|y_0| + y_0^2.$$

Define

$$d(y_0) = 9 - 6|y_0| + y_0^2.$$

This function is even, so it suffices to consider $y_0 \in [0,2]$. This quadratic is convex and

attains its minimum in $[0,2]$ at the boundary point $y_0 = 2$, where

$$d(2) = 4 - 12 + 9 = 1.$$

Therefore,

$$(3\sin t - y_0)^2 + (3\cos t)^2 \geq 1 \quad \text{for all } (y_0, t) \in [-2,2] \times [0,2\pi].$$

Consequently,

$$|f(y_0,t)| = \left| \frac{y_0 \sin t - 3}{(3\sin t - y_0)^2 + (3\cos t)^2} \right| \leq |y_0 \sin t - 3| \leq 5,$$

so $f \in L^1([-2,2] \times [0,2\pi])$. Thus, Fubini's theorem [Apostol 1969] allows interchange of the order of integration.

Set $P = y_0^2 + 9$ and $Q = 6y_0$. Then

$$I_C = \int_{-2}^{2} \left(\int_0^{2\pi} \frac{\sin t\, y_0 - 3}{P - Q\sin t}\, dt \right) dy_0.$$

Splitting the numerator $y_0 \sin t - 3$ into two terms,

$$I_C = \int_{-2}^{2} \left(\int_0^{2\pi} \frac{\sin t\, y_0 - 3}{P - Q\sin t}\, dt \right) dy_0 = \int_{-2}^{2} \left(\int_0^{2\pi} \frac{\sin t\, y_0}{P - Q\sin t}\, dt - \int_0^{2\pi} \frac{3}{P - Q\sin t}\, dt \right) dy_0.$$

To handle the first term, we rewrite $\frac{y_0 \sin t}{P - Q\sin t}$ by expressing $\sin t$ in terms of $P - Q\sin t$.

$$\frac{y_0 \sin t}{P - Q\sin t} = \frac{y_0}{Q}\left(\frac{P}{P - Q\sin t} - 1 \right) = \frac{y_0 P}{Q}\left(\frac{1}{P - Q\sin t} - \frac{1}{P} \right).$$

Substituting this identity yields

$$I_C = \int_{-2}^{2} \left(\left(\frac{y_0 P}{Q} - 3 \right) \int_0^{2\pi} \frac{dt}{P - Q\sin t} - \frac{2\pi y_0}{Q} \right) dy_0.$$

To evaluate the remaining integral, note that for $|P| > |Q|$,

$$\int \frac{dt}{P - Q\sin t} = \frac{2}{\sqrt{P^2 - Q^2}} \arctan\left(\frac{P\tan(\frac{t}{2}) - Q}{\sqrt{P^2 - Q^2}} \right),$$

a standard result [Gradshteyn and Ryzhik 2014]. We evaluate this anti-derivative on $[0,2\pi]$. Because the anti-derivative obtained from the substitution involves $\tan\left(\frac{t}{2}\right)$, which has a discontinuity at $t = \pi$, we split the integral as follows.

$$\int_0^{2\pi} \frac{dt}{P - Q\sin t} = \int_0^{\pi} \frac{dt}{P - Q\sin t} + \int_{\pi}^{2\pi} \frac{dt}{P - Q\sin t}.$$

Let $F(t)$ denote the antiderivative. The jump arises from the behavior of $\tan\left(\frac{t}{2}\right)$ at $t = \pi$.

Since

$$\lim_{t\to\pi^-}\tan\left(\frac{t}{2}\right)=+\infty, \qquad \lim_{t\to\pi^+}\tan\left(\frac{t}{2}\right)=-\infty$$

and

$$\lim_{x\to+\infty}\arctan(x)=\frac{\pi}{2}, \qquad \lim_{x\to-\infty}\arctan(x)=-\frac{\pi}{2}.$$

the jump in the antiderivative contributes

$$\mathrm{F}(\pi^-)-\mathrm{F}(\pi^+)=\pi.$$

Evaluating both pieces and accounting for this jump gives

$$\int_0^{2\pi}\frac{dt}{\mathrm{P}-\mathrm{Q}\sin t}=\frac{2\pi}{\sqrt{\mathrm{P}^2-\mathrm{Q}^2}}.$$

Substituting $\mathrm{P}=y_0^2+9$ and $\mathrm{Q}=6y_0$ gives

$$\mathrm{I_C}=\int_{-2}^{2}\left(\frac{y_0(y_0^2+9)}{6y_0}-3\right)\frac{2\pi}{\sqrt{(y_0^2+9)^2-(6y_0)^2}}-\frac{2\pi y_0}{6y_0}\,dy_0.$$

Observe that

$$\sqrt{(y_0^2+9)^2-(6y_0)^2}=\sqrt{(y_0^2-9)^2}=|y_0^2-9|.$$

Since $y_0\in[-2,2]$, we have $y_0^2-9<0$, and hence

$$\sqrt{(y_0^2+9)^2-(6y_0)^2}=9-y_0^2.$$

Although the simplification cancels a factor of y_0, the resulting expression extends continuously to $y_0=0$. We therefore interpret it via this continuous extension, so the simplification is valid. We obtain

$$\mathrm{I_C}=\int_{-2}^{2}\left(\frac{y_0^2-9}{6}\right)\frac{2\pi}{9-y_0^2}-\frac{\pi}{3}\,dy_0=\int_{-2}^{2}-\frac{2\pi}{3}\,dy_0=-\frac{8\pi}{3}.$$

Consequently,

$$3\mathrm{CI_C}=3\mathrm{C}\left(-\frac{8\pi}{3}\right)=-8\pi\mathrm{C}.$$

For any $\mathrm{C}\neq 0$, the circulation $\mathrm{I_{HK}}=-8\pi\mathrm{C}$ is non-zero. Since a conservative vector field must have zero circulation around every closed loop in Ω, the field $\mathbf{F}$ is not conservative, despite being curl-free everywhere on the domain. This completes the proof. □

7 Conclusion

The central question this paper addresses is concrete and classical: can a curl-free vector field fail to be conservative? The answer is yes, but only when the domain has a hole of the right kind. We have made this precise by constructing an explicit example on $\Omega = \mathbb{R}^n \setminus (\mathbb{R}^{n-2} \times \{0\} \times [-2,2])$, verifying that the field is smooth and curl-free on all of Ω, and confirming non-conservativeness by a direct computation that yields circulation $-8\pi C \neq 0$.

The result in this example is not surprising. It is well known that curl-free fields need not be conservative on non-simply connected domains. The value lies in the fact that the obstruction is made fully explicit. In the language of differential topology, our field represents a nontrivial cohomology class in $H^1_{dR}(\Omega)$. But to verify this, no cohomology theory is needed. The entire argument runs on undergraduate multivariable calculus, the fundamental group, and integration. The construction shows that a classical topological fact has a completely elementary method.

The geometry behind the example is worth taking note of. In $\mathbb{R}^n$ for $n \geq 3$, removing a set of codimension two is the minimal condition for producing a domain that is path-connected but not simply connected. Remove too little and loops can slip past the removed set and the domain remains simply connected and every curl-free field is conservative. Remove too much and the domain splits into disconnected pieces. Codimension two is the unique threshold at which non-trivial winding becomes possible, and $\mathbb{R}^{n-2} \times \{0\} \times [-2,2]$ is the cleanest higher-dimensional instance of this phenomenon.

We close with an honest accounting of what the construction does not do. The domain $\Omega = \mathbb{R}^n \setminus (\mathbb{R}^{n-2} \times \{0\} \times [-2,2])$ was chosen because its topology is as simple as possible while still producing the obstruction. These features were essential to keeping the circulation computation tractable. For a removed set of different geometry, such as a curved arc, a closed loop, or a higher-dimensional surface, the distributed singularity construction would need to be redesigned, and the resulting integrals may not evaluate in closed form. Similarly, the method as presented handles only domains with $\pi_1 \cong \mathbb{Z}$. A domain with a richer fundamental group , for instance, one with two independent non-contractible loops, would require a separate generating field for each generator, and the question of how these fields interact is not addressed here. Finally, while the construction produces one explicit non-conservative field, it does not characterise all such fields on Ω. A complete picture of the space of curl-free non-conservative fields, analogous to Foote's result [Foote 2018] in the planar case, remains open in higher dimensions.

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